NORTHWESTERN UNIVERSITY
SECOND SELECTION TEST

Thursday, October 7, 2004

NAME: ____________________

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem B1. Prove that there are no rational numbers u, v, w such that $u^2 + v^2 + w^2 = 7$.

Answer:

After clearing denominators we get the following Diophantine equation:

$$x^2 + y^2 + z^2 = 7t^2$$

where x, y, z, t are integers, and t is not zero. We may assume that $\gcd(x, y, z, t) = 1$ (otherwise divide them by their gcd.)

Next we study the equation modulo 8, noting that $7 \equiv -1 \pmod 8$:

$$x^2 + y^2 + z^2 + t^2 \equiv 0 \pmod 8 .$$

The only values of n^2 mod 8 are 0, 1 and 4. If the sum must be 0 mod 8 then none of the summands can be 1 mod 8. This implies that x, y, z, t are all even, which contradicts the hypothesis $\gcd(x, y, z, t) = 1$.

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Problem B2. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive numbers. Show that for all positive x,

$$(x+a_1)(x+a_2)\ldots(x+a_n) \leq \left(x + \frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n .$$

Answer:

By the Arithmetic-Geometric Mean Inequality:

$$\begin{aligned}\sqrt[n]{(x+a_1)(x+a_2)\ldots(x+a_n)} &\leq \frac{(x+a_1)+(x+a_2)+\cdots+(x+a_n)}{n} \\ &= x + \frac{a_1+a_2+\cdots+a_n}{n} .\end{aligned}$$

Raising both sides to the nth power we get the desired result.

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Problem B3. Let m be an odd positive integer. Prove that there is a positive integer n such that $2^n - 1$ is divisible by m.

Answer:

- *Solution 1*: (Using the Pigeonhole Principle.) Consider the $m+1$ numbers $2^0, 2^1, 2^2, \ldots, 2^m$. Since there are only m reminders modulo m, at least two of those numbers, say 2^t and 2^s ($t > s$), must have the same reminder modulo m, so $2^t - 2^s = 2^s(2^{t-s} - 1)$ is divisible by m. Since m is odd we have that $\gcd(m, 2^s) = 1$, so m must divide $2^{t-s} - 1$, and the result follows with $n = t - s$.

- *Solution 2*: (Using Euler's Theorem.) Euler's Theorem states that if a is an integer relatively prime with m then $a^{\phi(m)} \equiv 1 \pmod{m}$, where ϕ is Euler's function. Since m is odd then $\gcd(2, m) = 1$, hence $2^{\phi(m)} \equiv 1 \pmod{m}$, i.e., $2^{\phi(m)} - 1$ is a multiple of m.

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Problem B4.

(1) In a 120×150 rectangle (made out of unit squares joined along their sides), how many unit squares does its diagonal pass through?
(2) In a $120 \times 150 \times 180$ cuboid (made out of unit cubes joined along their faces), how many unit cubes does its diagonal pass through?
(Just "touching" at one point does not qualify as passing through).

Answer:

(1) Assume that the diagonal goes from $(0,0)$ to $(120,150)$. Its points will have coordinates $(x,y) = (120t, 150t)$, with $0 \le t \le 1$. As t goes from 0 to 1 the diagonal enters a unit square each time either x or y becomes an integer (except for $t = 1$), so the number of unit squares the diagonal goes through is the number of elements in the set

$$T = \{t \in [0,1) \mid 120t \in \mathbb{Z} \text{ or } 150t \in \mathbb{Z}\},$$

That set can be written as the union of the sets

$$T_1 = \{t \in [0,1) \mid 120t \in \mathbb{Z}\}, \quad T_2 = \{t \in [0,1) \mid 150t \in \mathbb{Z}\}.$$

By the Principle of Inclusion-Exclusion:

$$|T| = |T_1| + |T_2| - |T_1 \cap T_2| = 120 + 150 - \gcd(120,150) = \boxed{240}.$$

So the answer is 240.

(2) Assume that the diagonal goes from $(0,0,0)$ to $(120,150,180)$. Its points will have coordinates $(120t, 150t, 180t)$ with $0 \le t \le 1$. The number of unit cubes the diagonal goes through is the number of elements in the set

$$T = \{t \in [0,1) \mid 120t \in \mathbb{Z} \text{ or } 150t \in \mathbb{Z} \text{ or } 180t \in \mathbb{Z}\},$$

That set can be written as the union of the sets

$$T_1 = \{t \in [0,1) \mid 120t \in \mathbb{Z}\}, \quad T_2 = \{t \in [0,1) \mid 150t \in \mathbb{Z}\}, \quad T_3 = \{t \in [0,1) \mid 180t \in \mathbb{Z}\}.$$

By the Principle of Inclusion-Exclusion:

$$\begin{aligned}|T| &= |T_1| + T_2| + |T_3| - |T_1 \cap T_2| - |T_1 \cap T_3| - |T_2 \cap |T_3| + |T_1 \cap T_2 \cap T_3| \\ &= 120 + 150 + 180 - \gcd(120,150) - \gcd(120,180) - \gcd(150,180) \\ &\quad + \gcd(120,150,180) = \boxed{360}\end{aligned}$$

So the answer is 360.

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Problem B5. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and the product of any three elements of U is in U, show that at least one of the subsets T, U is closed under multiplication.

Answer:

Assume that none of T, U is closed under multiplication. Then there are two elements $t_1, t_2 \in T$ such that $t_1 t_2 = u_3 \in U$ and there are two elements $u_1, u_2 \in U$ such that $u_1 u_2 = t_3 \in T$. Consequently, the product $t_1 t_2 u_1 u_2 = t_1 t_2 t_3 = u_3 u_1 u_2$ is in both T and U, contradicting the assumption that T and U are disjoint.

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Problem B6. For positive integers n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2}\,,$$

as the $a_1, a_2, \dots, a_n$ range through all positive real values such that

$$a_1 + a_2 + \cdots + a_n = 17\,.$$

Find S_{10}.

Answer:

That sum is the length of a polygonal line connecting the points

$$(0,0),\ (1,a_1),\ (4,a_1+a_2),\ (9,a_1+a_2+a_3),\dots,\ (n^2,a_1+a_2+\cdots+a_n)\,.$$

For $n = 10$ the minimum value S_{10} is the length of a straight line connecting $(0,0)$ and $(100,17)$, i.e.,

$$S_{10} = \sqrt{100^2+17^2} = \boxed{\sqrt{10289}}\,.$$