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2007 NU PUTNAM DIAGNOSTIC EXAM

Instructions. In this test each problem has a solution that is an integer in the range from 0 to 999. You will find a problem on each sheet. Work the problem on each sheet explaining how you get your answer. Then **write your answer to each question on this page.** Your score will be the number of right answers.

Write your answers here.

1. ☐

2. ☐

3. ☐

4. ☐

5. ☐

6. ☐

7. ☐

8. ☐

9. ☐

10. ☐

11. ☐

12. ☐

13. ☐

14. ☐

15. ☐

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Problem A1. The sequence $a_1, a_2, \ldots, a_{98}$ satisfies $a_{n+1} = a_n + 1$ for $n = 1, 2, \ldots, 97$, and has sum 137. Find $a_2 + a_4 + a_6 + \cdots + a_{98}$.

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Problem A2. Find the smallest positive integer n such that every digit of $15n$ is 0 or 8.

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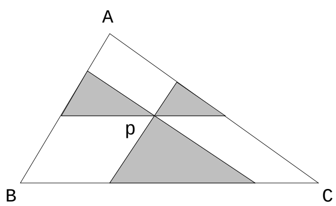
Problem A3. P is a point inside the triangle ABC. Lines are drawn through P parallel to the sides of the triangle. The areas of the three resulting triangles with a vertex at P have areas 4, 9 and 49. What is the area of ABC?



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Problem A4. A sequence of positive integers includes the number 68 and has arithmetic mean 56. When 68 is removed the arithmetic mean of the remaining numbers is 55. What is the largest number that can occur in the sequence?

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Problem A5. The reals x and y satisfy $\log_8 x + \log_4 (y^2) = 5$, and $\log_8 y + \log_4 (x^2) = 7$. Find xy.

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Problem A6. Three circles of radius 3 have centers at $P(14, 92)$, $Q(17, 76)$, and $R(19, 84)$. The line L passes through Q and the total area of the parts of the circles in each half-plane (defined by L) is the same. What is the absolute value of the slope of L.

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Problem A7. Let $\mathbb{Z}$ be the integers. The function $f : \mathbb{Z} \to \mathbb{Z}$ satisfies $f(n) = n - 3$ for $n \geq 1000$, and $f(n) = f(f(n+5))$ for $n < 1000$. Find $f(84)$.

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Problem A8. The equation $z^6 + z^3 + 1 = 0$ has a root $re^{i\theta}$, with $90^\circ < \theta < 180^\circ$. Find θ.

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Problem A9. The tetrahedron $ABCD$ has $AB = 3$, area $ABC = 15$, area $ABD = 12$, and the angle between the faces ABC and ABD is $30°$. Find its volume.

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Problem A10. An exam has 30 multiple-choice problems. A contestant who answers m questions correctly and n incorrectly (and does not answer $30-m-n$ questions) gets a score of $30+4m-n$. A contestant scores $N>80$. A knowledge of N is sufficient to deduce how many questions the contestant scored correctly. That is not true for any score M satisfying $80<M<N$. Find N.

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Problem A11. Three red counters, four green counters, and five blue counters are placed in a row in a random order. Let p/q be the probability that no two blue counters are adjacent written in lowest terms. What is $p+q$?

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Problem A12. Let $\mathbb{R}$ be the reals. The function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$, and for all x, $f(2+x) = f(2-x)$ and $f(7+x) = f(7-x)$. What is the smallest possible number of values x such that $|x| \leq 1000$ and $f(x) = 0$?

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Problem A13. Find $10\cot(\cot^{-1}3+\cot^{-1}7+\cot^{-1}13+\cot^{-1}21)$.

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Problem A14. What is the largest even integer that cannot be written as the sum of two odd composite positive integers?

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Problem A15. The real numbers x, y, z, w satisfy

$$\frac{x^2}{n^2-1^2}+\frac{y^2}{n^2-3^2}+\frac{z^2}{n^2-5^2}+\frac{w^2}{n^2-7^2}=1$$

for $n = 2, 4, 6, 8$. Find $x^2 + y^2 + z^2 + w^2$.