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Problem A1. Assume that a rectangle of dimensions a and b contains inside it another rectangle of dimensions a' and b'. Prove that $a' + b' < a + b$.

- *Answer:* The length of the diagonal of the inside rectangle will be less than that of the diagonal of the outside rectangle, hence $a'^2 + b'^2 < a^2 + b^2$. The same relation will hold for the areas: $a'b' < ab$. Hence $(a' + b')^2 = a'^2 + 2a'b' + b'^2 < a^2 + 2ab + b^2 = (a + b)^2$, and we get $a' + b' < a + b$.

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Problem A2. We place $4n$ points uniformly on a circle. Then we paint any $2n$ of them in red and the other $2n$ points in blue. Prove that regardless of which points we have painted with each color, there is always a straight line that divides the circle in half leaving exactly n red points and n blue points at each side.

- *Answer:* Pick a diameter. It divides the set of points into two parts each with $2n$ points. Pick one side of the diameter and assume there are r red points and b points at that side. Let d be the difference, $d = r - b$. Since $r + b = 2n$ is even, then r and b must have the same parity, so $d = r - b$ is also even. If $d = 0$ we are done, otherwise rotate the diameter slowly until completing a 180 degrees rotation. After the 180 degree rotation the difference between the number of red points and the number of blue points at the chosen side of the diameter will become $-d$, so changing its sign. On the other hand at the positions at which the diameter passes through a couple of points the difference between the number of red points and the number of blue points at the chosen side will change by 0 (if both points have the same color) or $+2$ or -2 (if they have different color). Consequently at some position of the diameter the difference must become zero, and that proves the statement.

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Problem A3. Show that for every positive integer n, $4^n + 6n - 1$ is a multiple of 9.

- *Answer:*

-*Solution 1:* The sequence 4^n mod 9 is periodic with period 3: $4^1 \equiv 4 \pmod 9$, $4^2 \equiv 7 \pmod 9$, $4^3 \equiv 1 \pmod 9$, $4^4 \equiv 4 \pmod 9$, etc. The same is true for $6n$ mod 9: $6 \cdot 1 \equiv 6 \pmod 9$, $6 \cdot 2 \equiv 3 \pmod 9$, $6 \cdot 3 \equiv 0 \pmod 9$, $6 \cdot 4 \equiv 6 \pmod 9$, etc. Hence, it suffices to prove the statement for $n = 1, 2, 3$:

$$\begin{aligned}
n = 1 \quad &\Rightarrow \quad 4^1 + 6 \cdot 1 - 1 = 4 + 6 - 1 = 9 \equiv 0 \pmod 9 \\
n = 2 \quad &\Rightarrow \quad 4^2 + 6 \cdot 2 - 1 = 16 + 12 - 1 = 27 \equiv 0 \pmod 9 \\
n = 3 \quad &\Rightarrow \quad 4^3 + 6 \cdot 3 - 1 = 64 + 18 - 1 = 81 \equiv 0 \pmod 9
\end{aligned}$$

-*Solution 2:* By induction. For $n = 1$ we have $4^n + 6n - 1 = 4 + 6 - 1 = 9$, which is a multiple of 9. Next assume $4^n + 6n - 1$ is a multiple of 9. We must prove that $4^{n+1} + 6(n+1) - 1$ is also a multiple of 9. In fact, $4(4^n + 6n - 1) = 4^{n+1} + 24n - 4$ will be a multiple of 9, and we have that $18n - 9$ is a multiple of 9, hence:

$$4^{n+1} + 6(n+1) - 1 = (4^{n+1} + 24n - 4) - (18n - 9)$$

is a multiple of 9, and this completes the induction step.

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Problem A4. Prove $\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx \geq \frac{\pi}{2}(e-1)$.

- *Answer:* Because of concavity of $\sin x$ in $[0, \frac{\pi}{2}]$ we have $\sin x \geq \frac{2}{\pi}x$ for $0 \leq x \leq \frac{\pi}{2}$, hence

$$\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx \geq \int_0^{\frac{\pi}{2}} e^{\frac{2}{\pi}x}\,dx = \frac{\pi}{2}\left[e^{\frac{2}{\pi}x}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}(e-1)\,.$$

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Problem A5. A number n has 250 positive divisors, sorted and indexed in increasing order: $1 = d_1 < d_2 < d_3 < \cdots < d_{250} = n$. Ted is allowed to pick two indices i and j $(1 \leq i, j \leq 249)$, with the condition that $i + j \neq 251$, and he is given in return divisors d_i and d_j. Show that Ted can always find the value of n by picking appropriately those two indices.

- *Answer:* We prove that knowing d_{248} and d_{249} is enough to determine the value of n.

- *Case 1:* If d_{248} does not divide d_{249} then $\operatorname{lcm}(d_{248}, d_{249})$ is a divisor of n greater than d_{249}, hence $\operatorname{lcm}(d_{248}, d_{249}) = d_{250} = n$.

- *Case 2:* If d_{248} divides d_{249} then $d_{249} = kd_{248}$ for some integer $k > 1$. Using $d_j d_{251-j} = n$ we get $d_3 = \frac{n}{d_{248}} = \frac{kn}{d_{249}} = kd_2 > k$. It follows that k is a divisor of n satisfying $1 = d_1 < k < d_3$, hence $k = d_2$, and $n = d_2 d_{249} = kd_{249}$.