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FALL 2017 NU PUTNAM SELECTION TEST

Problem A1. Prove that the following equation has no solutions in positive integers:

$$8x^4 + 4y^4 + 2z^4 = t^4 \,.$$

(Hint: t must be an even integer.)

- *Answer:* Since the left hand side is even then t must be even, $t = 2t'$, hence:

$$8x^4 + 4y^4 + 2z^4 = 16t'^4 \,,$$

and simplifying by 2:

$$4x^4 + 2y^4 + z^4 = 8t'^4 \,,$$

Now all terms different from z^4 are even, hence z must be even: $z = 2z'$, hence:

$$4x^4 + 2y^4 + 16z'^4 = 8t'^4 \,.$$

Simplifying by 2 again we get

$$2x^4 + y^4 + 8z'^4 = 4t'^4 \,.$$

A similar reasoning shows that y must be even, $y = 2y'$, and after simplifying we get:

$$x^4 + 8y'^4 + 4z'^4 = 2t'^4 \,.$$

Next we do the same for x so we have $x = 2x'$, and after simplifying:

$$8x'^4 + 4y'^4 + 2z'^4 = t'^4 \,.$$

This shows that given a solution in positive integers (x, y, z, t) then there is another solution (x', y', z', t') in positive integers with $x' = x/2 < x$, $y' = y/2 < y$, $z' = z/2 < z$, $t' = t/2$. Repeating the reasoning we get an infinite sequence of positive solutions of the form $(x/2^k, y/2^k, z/2^k, t/2^k)$ for k arbitrarily large, but that is impossible because no integer is divisible by arbitrarily large powers of 2.

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Problem A2. Let $a_1, a_2, a_3, \dots$ a strictly increasing sequence of positive integers, i.e., $a_n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, and $n < m \Rightarrow a_n < a_m$ for every m, n. Find all strictly increasing functions functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$, where $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$, such that $f(a_n) \leq a_n$ for every $n \in \mathbb{Z}^+$.

- *Answer:* The only solution is $f(n) = n$ for every $n \in \mathbb{Z}^+$.

In fact, since f is strictly increasing then $f(n) < f(n+1)$ for every n, hence $f(n) + 1 \leq f(n+1)$, and by induction $f(n) = f(1 + (n-1)) \geq f(1) + n - 1$. By hypothesis we have $f(a_1) \leq a_1$, hence $f(1) + a_1 - 1 \leq f(a_1) \leq a_1$, and from here we get $f(1) \leq 1$. But $f(1) \in \mathbb{Z}^+$, hence $f(1) = 1$, and $f(n) \geq n$.

Finally we show that $f(n) > n$ cannot hold for any n. In fact if $f(n) > n$ for some n, consider any $a_k > n$. Then $f(a_k) = f(n + (a_k - n)) \geq f(n) + a_k - n > a_k$, contradicting the hypothesis $f(a_n) \leq a_n$ for every n.

So, we have $f(n) \geq n$, and $f(n) \not> n$, hence $f(n) = n$ for every n, Q.E.D.

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Problem A3. Find the following limit:

$$L = \lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} \left(1+\frac{k}{n}\right)^{1/(1+\frac{k}{n})}} .$$

(Hint: Take the logarithm of the expression under the limit.)

- *Answer:* The logarithm of the expression inside the limit is

$$\sum_{k=1}^{n} \frac{\log\left(1+\frac{k}{n}\right)}{1+\frac{k}{n}} \frac{1}{n} .$$

That is a Riemann sum for the integral

$$\int_0^1 \frac{\log(1+x)}{1+x}\, dx = \left[\frac{1}{2}\log^2(1+x)\right]_0^1 = \frac{\log^2 2}{2} .$$

Hence the limit is

$$L = e^{\frac{\log^2 2}{2}} = 2^{\frac{\log 2}{2}} .$$

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Problem A4. A fair coin is tossed repeatedly. What is the expected number of times the coin will be tossed until getting two heads in a row for the first time?

- *Answer:* Let E the expected number of tosses until getting two heads in a row for the first time.

Denote heads and tails as H and T respectively. The following three things can happen:

(1) We get T in the first toss, with probability $1/2$.
(2) We get HT in the first two tosses, with probability $1/4$.
(3) We get HH in the first two tosses, with probability $1/4$.

In cases (1) and (2), after getting any of those results, the expected number of additional tosses we must wait to get two heads in a row is again E, which yields $1+E$ tosses with probability $1/2$, and $2+E$ with probability $1/4$. In case (3) we end the sequence after only 2 tosses, and this happens with probability $1/4$. Hence:

$$E = \frac{1}{2}(1+E) + \frac{1}{4}(2+E) + \frac{1}{4}\cdot 2\,.$$

From here we get $E = 6$,

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Problem A5. Let a_k, $k = 1, 2, 3, \ldots$, be a sequence of strictly positive numbers of period $2N$. Show that

$$\sum_{j=1}^{2N} \frac{a_{N+j}}{a_j} \geq 2N\,.$$

- *Answer:* We will use $x + \frac{1}{x} \geq 2$ for every positive real number x. We have

$$\begin{aligned}
\sum_{j=1}^{2N} \frac{a_{N+j}}{a_j} &= \sum_{j=1}^{N} \frac{a_{N+j}}{a_j} + \sum_{j=1}^{N} \frac{a_{2N+j}}{a_{N+j}} \\
&= \sum_{j=1}^{N} \frac{a_{N+j}}{a_j} + \sum_{j=1}^{N} \frac{a_j}{a_{N+j}} \qquad (a_{2N+j} = a_j) \\
&= \sum_{j=1}^{N} \left(\frac{a_{N+j}}{a_j} + \frac{a_j}{a_{N+j}} \right) \\
&\geq \sum_{j=1}^{N} 2 = 2N\,.
\end{aligned}$$

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Problem A6. Given any positive integer a consider the sequence $a_n = a^{a^{a^n}}$, $n = 1, 2, 3, \ldots$. Prove that regardless of the integer a chosen, the rightmost digit of the decimal representation of a_n remains constant.

- *Answer:* First note that if $a = 1$ then $a_n = 1$ for every n, and this case is trivial, so in the following we assume $a \geq 2$.

We must prove that a_n is constant modulo 10.

Note that for any integer a, the sequence a^k for $k = 1, 2, 3, \ldots$ is periodic modulo 10 with a period of 1, 2 or 4, with the rightmost digits following one of these patterns:

$$0 \to 0$$
$$1 \to 1$$
$$2 \to 4 \to 8 \to 6 \to 2$$
$$3 \to 9 \to 7 \to 1 \to 3$$
$$4 \to 6 \to 4$$
$$5 \to 5$$
$$6 \to 6$$
$$7 \to 9 \to 3 \to 1 \to 7$$
$$8 \to 4 \to 2 \to 6 \to 8$$
$$9 \to 1 \to 9$$

Hence a^k modulo 10 depends only on the value of k modulo 4, meaning that if $k \equiv k' \pmod 4$ then $a^k \equiv a^{k'} \pmod{10}$.

Hence a^{a^k} modulo 10, depends only on the value of a^k modulo 4. But a^k is *eventually* periodic modulo 4 with period 1 or 2, and patterns $0 \to 0$, $1 \to 1$, $2 \to 0 \to 0$, $3 \to 1 \to 3$ (all modulo 4). Furthermore note that we can drop "eventually" if $k \geq 2$, since the only case in which a^k is not strictly periodic module 4 is at the starting point of the pattern $2 \to 0 \to 0$, more specifically, if $a \equiv 2 \pmod 4$ then $a^1 \equiv 2 \pmod 4$, and for $k \geq 2$ we have $a^k \equiv 0 \pmod 4$. Hence for $k \geq 2$, a^k modulo 4 depends only on the parity of k, meaning than if $k, k' \geq 2$ and k and k' have the same parity then $a^k \equiv a^{k'} \pmod 4$.

Finally we have that the parity of a^n is the same as the parity of a, hence the parity of a^n remains constant. This fact (combined with $a^n \geq 2$) implies that a^{a^n} modulo 4 remains constant. And this implies that $a^{a^{a^n}}$ modulo 10 remains constant. $\square$